Sections 8.2-8.4 parts 2&3

Chapter 8 Normal Subgroups

<u>Def</u> A subgroup $N \subset G$ such that $Na = aN$ for every $a \in G$ is called <u>normal</u>.

$$G = \bigcup_{a\in G} Na = \bigcup_{a \in G} aN$$ left and right cosets are the same

The set of cosets is denoted by G/N } In ring theory, that was R/I for an ideal I

Group structure - operation - on G/N

<u>Def</u> $(Na)\cdot(Nc) = Nac$

(the coset whose representative is a)(the coset whose representative is c)

= (the coset whose representative is ac)

Th 8.10 This operation is well-defined:
8.12 For any $b \in Na$ and $d \in Nc$, we have $Nbd = Nac$

<u>Pf</u>

Since the cosets are equivalence classes, it suffices to check <u>$bd \in Nac$</u>

$b \in Na$ means $b = n_1 a$ with $n_1 \in N$

$d \in Nc$ means $d = n_2 c$ with $n_2 \in N$

$bd = n_1 a n_2 c$ It may be that $an_2 \neq n_2 a$

However, $aN = Na$ – the subgroup N is normal

$an_2 \in aN = Na$ means $\underline{an_2 = n_3 a}$ | such $n_3 \in N$ exists

$\underline{bd} = n_1 a n_2 c = n_1 n_3 a c \in \underline{Nac}$ because $n_1 n_3 \in N$ as soon as both $n_1 \in N$, $n_3 \in N$

Th 8,13 (1) G/N with this operation is a group

Pf – one checks with group axioms } the identity is the coset of the identity in G

$N = Ne_G = e_G N$

Terminology

G/N is called factor-group
quotient group

Characterizations of normality

Th 8.11 A subgroup $N \subset G$ of a group G is normal if and only if:

(2) $a^{-1}Na \subseteq N$ for every $a \in G$ $\quad a^{-1}Na = \{a^{-1}na \mid n \in N\}$

(3) $aNa^{-1} \subseteq N$ ——— " ———

(4) $a^{-1}Na = N$ ——— " ———

(5) $aNa^{-1} = N$

Ex 3 p 252. Ex 15 p 253

Ex 23 p 254

Every subgroup of index 2 is normal.

$G \supset N \qquad [G:N]=2$

$G = \bigcup_{a\in G} Na = N \cup Na; \qquad N \cap Na = \phi$

$Na = \{ b \in G \mid b \notin N \}$ For every $b \in Na$, $Na = Nb$

We'll prove 8.11 (2):

for every $a \in G$, we have $a^{-1} N a \subseteq N$

If $a \in N$, then $a^{-1} n a \in N$ for every $n \in N$ because N is a subgroup.

Let $\underline{a \notin N}$. Then $G = N \cup Na \qquad N \cap Na = \phi$

For the sake of a contradiction, assume that $\underline{a^{-1} n a \notin N}$ for some $n \in N$.

Then $a^{-1} n a \in Na$

$a^{-1} n a = n_1 a$ with $n_1 \in N$

$a^{-1} n = n_1 \qquad a^{-1} = n_1 n^{-1} \qquad \underline{a} = n n_1^{-1} \underline{\in N}$, a contradiction.